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Multi-Field Generalisations of the Klein-Gordon Theory associated with p-Branes

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Abstract

The purpose of this article is to initiate a study of a class of Lorentz invariant, yet tractable, Lagrangian Field Theories which may be viewed as an extension of the Klein-Gordon Lagrangian to many scalar fields in a novel manner. These Lagrangians are quadratic in the Jacobians of the participating fields with respect to the base space co-ordinates. In the case of two fields, real valued solutions of the equations of motion are found and a phenomenon reminiscent of instanton behaviour is uncovered; an ansatz for a subsidiary equation which implies a solution of the full equations yields real solutions in three-dimensional Euclidean space. Each of these is associated with a spherical harmonic function.

1 Introduction

Recently a class of field theories which arise as a continuation of the Dirac-Born-Infeld theories to the case where the dimension of the base space is larger than that of the target space was investigated [1, 2, 3]. We named these theories Companion theories, as they appeared as associated with string and D-brane theories, and the corresponding Lagrangians as Companion Lagrangians, which were of square-root form,

$$\mathcal{L} = \sqrt{\det \left| \frac{\partial \phi^i}{\partial x^\mu} \frac{\partial \phi^j}{\partial x^\mu} \right|}\,, \tag{1}$$

where the set of functions ϕ^i $(i = 1, 2, \cdots, p+1)$ each dependent upon co-ordinates x^μ $(\mu = 1, 2, \cdots, d > p+1)$ is a mapping from the base space to the target space.

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A remarkable property of the theory is that the equations of motion derived from $\mathcal{L}$ are invariant under arbitrary field redefinitions $\phi^i \to \phi'^i(\phi^j)$. In consequence, a large class of solutions may be found, and these equations are completely integrable when $d = p + 2$. However the solutions are for the most part given in implicit form.

In this article, we shall instead study the Lagrangian density which is the square of (1), for which it will turn out that it inherits much of the integrability of the previous case, but permits solutions of the equations of motion in explicit form. Indeed, while this investigation was motivated by the notion of the Companion Lagrangian, it may be regarded as a study in its own right of a natural generalisation of the Klein-Gordon Lagrangian to many fields obtained by replacing $\partial\phi/\partial x_\mu$ by the Jacobian $J_{\mu_1\mu_2\cdots\mu_{p+1}}$ of $p+1$ fields with respect to a set of the co-ordinates, and summing over all combinations of $J^2_{\mu_1\mu_2\cdots\mu_{p+1}}$.

After a short description of the equations of motion for the general case of $p+1$ fields in d dimensions, we analyse the specific example of two fields. We also give attention to a first order differential equation as an ansatz to solve the full equations of motion with $p+1$ fields in $d = p+2$ dimensional Euclidean space. In the simplest, non-trivial example among such theories, corresponding to two fields in three dimensions, we explicitly demonstrate a large class of solutions, including a solution of the ansatz equation associated with each spherical harmonic function.

In this article, we shall restrict ourselves in the main to the massless case. The example studied should not be confused despite analogous duality features with the study of antisymmetric tensor fields, as in Freedman and Townsend [4]. Here the antisymmetric field has an additional structure, as it is constructed from two scalar fields.

2 Companion Equations

2.1 Notation

We work in d-dimensional flat space with the totally antisymmetric tensor $\epsilon_{\nu_1\nu_2\ldots\nu_d}$ with $\epsilon_{12\ldots d} = +1$. Indices with an arrow above them denote a set of several indices. $\vec{\mu}, \vec{\nu}, \vec{\rho}$ each have $p+1$ components, e.g., $\vec{\nu} = \{\nu_1, \nu_2, \ldots, \nu_{p+1}\}$. $\vec{\tau}, \vec{\kappa}, \vec{\eta}$ each have $(d-p-1)$ components, e.g., $\vec{\kappa} = \{\kappa_1, \kappa_2, \ldots, \kappa_{d-p-1}\}$. If the prime $'$ is used for the indices, $\vec{\mu}, \vec{\nu}$ or $\vec{\tau}, \vec{\kappa}$, then their components start from the second entry of un-primed ones, e.g., $\vec{\nu}' = \{\nu_2, \ldots, \nu_{p+1}\}$ and $\vec{\kappa}' = \{\kappa_2, \ldots, \kappa_{d-p-1}\}$.

2.2 Jacobians

We define Jacobians $\tilde{J}_{\vec{\kappa}}$ and their duals $J_{\vec{\nu}}$ as

$$\begin{aligned}\tilde{J}_{\vec{\kappa}} = \tilde{J}_{\kappa_1\kappa_2\ldots\kappa_{d-p-1}} &= \epsilon_{\kappa_1\kappa_2\ldots\kappa_{d-p-1}\nu_1\nu_2\ldots\nu_{p+1}}\phi^1_{\nu_1}\phi^2_{\nu_2}\ldots\phi^{p+1}_{\nu_{p+1}} \\ &= \frac{1}{(p+1)!}\,\epsilon_{\vec{\kappa}\vec{\nu}}\,\epsilon_{i_1\ldots i_{p+1}}\phi^{i_1}_{\nu_1}\ldots\phi^{i_{p+1}}_{\nu_{p+1}} \equiv \frac{1}{(p+1)!}\,\epsilon_{\vec{\kappa}\vec{\nu}}\,J_{\vec{\nu}}\ ,\end{aligned} \tag{2}$$

where $\phi^i_\nu = \partial\phi^i/\partial x^\nu$. The derivatives of the Jacobians are

$$\frac{\partial\tilde{J}_{\vec{\kappa}}}{\partial\phi^i_\mu} = \frac{1}{p!}\,\epsilon_{\vec{\kappa}\mu\vec{\nu}'}\,\epsilon_{ii_2\ldots i_{p+1}}\phi^{i_2}_{\nu_2}\ldots\phi^{i_{p+1}}_{\nu_{p+1}} \equiv \frac{1}{p!}\,\epsilon_{\vec{\kappa}\mu\vec{\nu}'}\,J_{i,\vec{\nu}'}\ . \tag{3}$$

Using the Jacobians, the Companion Lagrangian $\mathcal{L}_1$ without square-root is written as

$$\mathcal{L}_1 \;=\; \det\left|\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial\phi^j}{\partial x^\mu}\right| \;=\; \frac{1}{(d-p-1)!}\,\tilde{J}_{\vec{\kappa}}\tilde{J}_{\vec{\kappa}} \;=\; \frac{1}{(p+1)!}\,J_{\vec{\mu}}J_{\vec{\mu}}\ . \tag{4}$$

The equation of motion for $\mathcal{L}_1$ is then given by

$$\partial_\mu\left(\frac{\partial\mathcal{L}_1}{\partial\phi^i_\mu}\right) = \frac{2}{(d-p-1)!}\,\frac{\partial\tilde{J}_{\vec{\kappa}}}{\partial\phi^i_\mu}\frac{\partial\tilde{J}_{\vec{\kappa}}}{\partial\phi^j_\nu}\,\phi^j_{\mu\nu} = \frac{2}{p!}\,J_{i,\vec{\nu}'}\,\partial_\mu J_{\mu\vec{\nu}'} = 0\ . \tag{5}$$

3 Solutions of Quadratic Jacobian Lagrangian

Let us consider the specific case of two fields, in which the Jacobians are given, with $\phi^1 = u$ and $\phi^2 = v$,

$$J_{\mu\nu} \;=\; \frac{\partial u}{\partial x^\mu}\frac{\partial v}{\partial x^\nu} \;-\; \frac{\partial u}{\partial x^\nu}\frac{\partial v}{\partial x^\mu}\ . \tag{6}$$

It is noteworthy that $J_{\mu\nu}$ is like a field strength $F_{\mu\nu}$ for the gauge potential

$$A_\mu \;=\; u\frac{\partial v}{\partial x^\mu} + \frac{\partial w}{\partial x^\mu}\ , \tag{7}$$

where the field w represents an ambiguity by gauge transformation. This is the Clebsch representation for a vector field in the case of three dimensions [2, 5]. Using subscripts to denote derivatives, e.g. $u_\mu = \partial u/\partial x^\mu$, the equations of motion (5) in this case are simply

$$u_\nu\frac{\partial}{\partial x^\mu}J_{\mu\nu} \;=\; 0\ , \qquad v_\nu\frac{\partial}{\partial x^\mu}J_{\mu\nu} \;=\; 0\ . \tag{8}$$

Note that these equations mean that the vector $\partial J_{\mu\nu}/\partial x^\mu$ is orthogonal to both the vectors u_ν and v_ν, and this implies the following equation in three dimensions,

$$\frac{\partial}{\partial x^\mu}(u_\mu v_\nu - u_\nu v_\mu) \;=\; \lambda(u,v)\,\epsilon_{\nu\rho\sigma}u_\rho v_\sigma \;, \tag{9}$$

where λ is a function of $u,\ v$.

There is a large class of solutions which may be simply categorised as follows,

$$u \;=\; f(a_\rho x^\rho,\ b_\rho x^\rho)\;, \qquad v \;=\; g(a_\rho x^\rho,\ b_\rho x^\rho)\;, \tag{10}$$

where $f,\ g$ are arbitrary functions of two variables and the constant vectors $a_\rho,\ b_\rho$ satisfy

$$(a_\rho a_\rho)(b_\sigma b_\sigma) \;-\; (a_\rho b_\rho)^2 \;=\; 0\;. \tag{11}$$

Notice that in Euclidean space such solutions, if they are to be nontrivial, are necessarily complex. The trivial solution $a_\rho = \lambda b_\rho$ implies that u and v are functionally related. This result is a direct generalisation of the Klein-Gordon situation, that $\phi(\vec{a}\cdot\vec{x})$ is a solution of the massless equation iff $\vec{a}$ is null. A specific example of this form of solution in three dimensions is given by

$$u \;=\; f(x + i\sin\theta y, z + i\cos\theta y)\;, \qquad v \;=\; g(x + i\sin\theta y, z + i\cos\theta y)\;. \tag{12}$$

In Minkowski space, of course, real solutions may be obtained by setting $t = iy$. In fact a yet more general construction can be given; in three dimensions it is

$$u \;=\; F(a_\rho x^\rho,\ b_\rho x^\rho,\ c_\rho x^\rho)\;, \qquad v \;=\; G(a_\rho x^\rho,\ b_\rho x^\rho,\ c_\rho x^\rho)\;, \tag{13}$$

where $F,\ G$ are arbitrary functions of three variables and the constant vectors $a_\rho,\ b_\rho,\ c_\rho$ satisfy

$$(a_\rho a_\rho)(b_\sigma b_\sigma) \;-\; (a_\rho b_\rho)^2 \;=\; 0\;, \qquad c_\rho \;=\; \epsilon_{\rho\tau\sigma}a_\tau b_\sigma\;, \tag{14}$$

while in higher dimensions the definition of c_ρ is replaced by

$$c_\rho \;=\; \epsilon_{\rho\lambda_1\lambda_2\ldots\tau\sigma}\,k^1_{\lambda_1}k^2_{\lambda_2}\ldots a_\tau\, b_\sigma\;, \tag{15}$$

and $\vec{k}^1\ \vec{k}^2\ldots$ are constant vectors orthogonal to $\vec{a}$ and $\vec{b}$. There is an alternative strategy in seeking solutions, which we elucidate below.

4 Ansatz for Companion Equations

In this section, we consider $p+1$ fields in $p+2$ dimensional Euclidean space and obtain auxiliary equations whose solution implies that of the full equations of motion in the manner of the equations of self-duality.

4.1 Ansatz equations for $(d = p + 2, p + 1)$

In the $d = p + 2$ dimensional case, the Jacobian defined in (2) has one index κ; $\tilde{J}_\kappa = \frac{1}{(p+1)!}\epsilon_{\kappa\nu_1\ldots\nu_{p+1}} J_{\nu_1\ldots\nu_{p+1}}$. We make the assumption that the Jacobian is expressed as the derivative of a function $G(x)$,

$$\tilde{J}_\kappa = \frac{1}{(p+1)!}\,\epsilon_{\kappa\nu_1\ldots\nu_{p+1}} J_{\nu_1\ldots\nu_{p+1}} = \frac{1}{(p+1)!}\,\epsilon_{\kappa\vec{\nu}}\, J_{\vec{\nu}} = \partial_\kappa G(x) \ , \tag{16}$$

or equivalently, in terms of the dual $J_{\vec{\nu}}$,

$$J_{\vec{\nu}} = \epsilon_{\kappa\vec{\nu}}\,\partial_\kappa G(x) \ . \tag{17}$$

From (17), we can easily see solutions of the ansatz to satisfy the Companion equation of motion (5), since $\partial_{\nu_1} J_{\nu_1\vec{\nu}'} = \epsilon_{\kappa\nu_1\vec{\nu}'}\partial_{\nu_1}\partial_\kappa G(x) = 0$. Differentiating the LHS of (16) and using the Jacobi identity for $J_{\vec{\nu}}$, $\partial_{[\kappa} J_{\vec{\nu}]} = 0$, we obtain the harmonic equation for $G(x)$,

$$\Box G(x) = \partial_\mu\partial_\mu G(x) = 0 \ . \tag{18}$$

The relation (17) between $J_{\vec{\nu}}$ and $\partial_\kappa G(x)$ may be considered analogous to the implications of self-duality in monopole theory. The Jacobi identity for the former gives the harmonic equation for the latter, while the integrability condition for the latter leads to a divergence-free equation for the former which solves the Companion equations (5).

4.2 $(d = 3, p = 1)$ case

Let us apply the ansatz equation (16) for the case with two fields in three dimensions,

$$\tilde{J}_\mu = \frac{1}{2}\,\epsilon_{\mu\nu\sigma} J_{\nu\sigma} = \epsilon_{\mu\nu\sigma}\, u_\nu\, v_\sigma = \partial_\mu G(x) \ . \tag{19}$$

In [6], this equation has appeared as a dual version of the Poisson Bracket version of the three-dimensional Nahm equations, for which Ward has given an implicit solution [7]. From (19), we see that the vector $\partial_\mu G$ is orthogonal to both vectors u_ν and v_σ. As is well known, with the use of polar co-ordinates

$$x = r\sin\theta\cos\phi \ , \quad y = r\sin\theta\sin\phi \ , \quad z = r\cos\theta \ , \tag{20}$$

the 3D harmonic equation for G is solved in terms of spherical harmonic functions,

$$G = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_l^m\, G_l^m \ , \qquad G_l^m = \frac{1}{r^{l+1}}\, Y_l^m(\theta,\phi) \ ,$$
$$Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\theta)\, e^{im\phi} \ , \tag{21}$$

where a_l^m are constants and $P_l^{|m|}(\cos\theta)$ are associated Legendre polynomials.

Let us first apply each G_l^m to the ansatz equation and find corresponding solutions of u and v. The homogeneity of the RHS of (19) is of degree $-(l+2)$, as G_l^m is of degree $-(l+1)$, Thus, at a first attempt to solve the ansatz, it is natural to impose a homogeneity condition upon both functions u and v,

$$u(tx,ty,tz) = t^{-k}\,u(x,y,z)\ , \qquad v(tx,ty,tz) = t^{-n}\,v(x,y,z)\ , \tag{22}$$

or equivalently,

$$x^\mu\partial_\mu u(x,y,z) = -k\,u(x,y,z)\ , \qquad x^\mu\partial_\mu v(x,y,z) = -n\,v(x,y,z)\ . \tag{23}$$

When u and v are expressed in terms of $\{r,\theta,\phi\}$, their r dependences are determined by the condition (22) as $u = r^{-k}U(\theta,\phi)$ and $v = r^{-n}V(\theta,\phi)$. The degrees of u, v and G are obviously related by

$$k + n = l\ . \tag{24}$$

We suppose the numbers k and n to be integers and also $k \geq n \geq 0$ to make sure that u and v are well behaved as $r \to \infty$.

In the $l = 0$ case, $G = G_0^0 = 1/r$ and functions u and v are homogeneous of degree zero. Then a solution of the ansatz (19) is easily obtained as

$$u = \cos\theta = \frac{z}{r}\ , \qquad v = \phi = \arctan\frac{y}{x}\ . \tag{25}$$

For the $l > 0$ case, multiplying (19) by $\epsilon_{\mu\lambda\rho}x_\lambda$, with the use of the homogeneity condition of u and v, we have

$$-\,k\,u\,v_\rho + n\,v\,u_\rho = \epsilon_{\mu\lambda\rho}x_\lambda G_\mu\ . \tag{26}$$

Conversely, acting with the operator $\epsilon_{\tau\sigma\rho}\partial_\sigma$ upon (26), we have $l(\epsilon_{\tau\sigma\rho}u_\sigma v_\rho - G_\tau) = x_\tau \triangle G$, showing the equivalence between (19) and (26), provided $G(x)$ is a solution of the harmonic equation.
Let us consider the canonical transformation,

$$u' \;=\; -\frac{1}{l}u\,v\ , \qquad v' \;=\; \ln\frac{v^k}{u^n}\ , \qquad \{u',v'\}_{(u,v)} = 1\ , \tag{27}$$

under which the ansatz equation (19) is invariant. Then (26) is transformed into the form of the equation (26) with $(k,n) = (l,0)$. Therefore, in the following, we concentrate on the $(l,0)$-type solutions.

For the $(l,0)$ case, the equation (26) becomes

$$v_\mu = \alpha(x)\,\epsilon_{\mu\lambda\rho}x_\lambda G_\rho = l_\mu G\ , \qquad \alpha(x) = \frac{1}{l\,u}\ , \tag{28}$$

where l_μ are angular momentum operators, which are rewritten in terms of $\{r, \theta, \phi\}$,

$$l_1 = -\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi} \ , \quad l_2 = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi} \ , \quad l_3 = \frac{\partial}{\partial\phi} \ . \quad (29)$$

The first equation in (28) can be recast as

$$v_\theta = -\frac{\alpha(x)\, r}{\sin\theta}\, G_\phi \ , \qquad v_\phi = \alpha(x)\, r \sin\theta\, G_\theta \ . \tag{30}$$

The integrability condition $\partial_\phi v_\theta = \partial_\theta v_\phi$ gives a first order equation for $\alpha(x)$,

$$\frac{G_\phi}{\sin^2\theta}\,\partial_\phi \ln\alpha + G_\theta\,\partial_\theta \ln\alpha = -\frac{1}{\sin^2\theta}(\sin\theta\,\partial_\theta(\,\sin\theta G_\theta) + G_{\phi\phi}) = -\vec{l}^{\,2}\, G \ , \tag{31}$$

where the operator $\vec{l}^{\,2} = l_1^2 + l_2^2 + l_3^2$.

For $G = G_l^m$ $(l = 1, 2, \cdots, m = 0, 1, \cdots, l)$ in (21), the operators $\vec{l}^{\,2}$ and $\partial_\phi = l_3$ give eigenvalues $-l(l+1)$ and im, respectively. Then (31) becomes, with $q = \cos\theta$

$$im\frac{P_l^m(q)}{\sin^2\theta}\,\partial_\phi \ln\alpha + \sin^2\theta\,\frac{dP_l^m(q)}{dq}\,\partial_q \ln\alpha = l(l+1)\,P_l^m(q) \ . \tag{32}$$

To find real solutions for $\alpha(x)$, we assume $\alpha(x) = \alpha(r, q)$, which gives

$$\partial_q \ln\alpha(r, q) = \frac{l(l+1)P_l^m}{(1-q^2)(dP_l^m/dq)} \ . \tag{33}$$

The function u is then given as

$$u = \frac{1}{l\alpha} = \frac{e^{-T(q)}}{lr^l} \ , \qquad T(q) = l(l+1)\int dq\,\frac{P_l^m(q)}{(1-q^2)dP_l^m(q)/dq} \ . \tag{34}$$

The form of v is easily evaluated by the second equation of (30), for $m \neq 0$,

$$v = \frac{i}{m}\,e^{T(q)}\,(1-q^2)\frac{dP_l^m(q)}{dq}\,e^{im\phi} \ . \tag{35}$$

We can easily check this solution to satisfy the first equation for v_θ in (30) by using the fact that the associated Legendre polynomials satisfy

$$\frac{d}{dq}((1-q^2)\frac{d}{dq}P_l^m(q)) + (l(l+1) - \frac{m^2}{1-q^2})P_l^m(q) = 0 \ . \tag{36}$$

Having chosen the function u to be real, we obtain solutions for G_l^{-m} $(m = 1, \cdots, l)$ by taking the complex conjugation of (35),

$$u = \frac{e^{-T(q)}}{lr^l} \ , \qquad v = -\frac{i}{m}\,e^{T(q)}\,(1-q^2)\frac{dP_l^m(q)}{dq}\,e^{-im\phi} \ . \tag{37}$$

Taking a linear combination of solutions for G_l^m and G_l^{-m}, we have real solutions for the ansatz equation,

$$
\begin{aligned}
G &= \frac{P_l^m(q)}{r^{l+1}}(a\cos m\phi + b\sin m\phi)\ , \qquad (m = 1, \cdots, l) \\
u &= \frac{e^{-T(q)}}{lr^l}\ , \quad v = \frac{e^{T(q)}}{m}\,(1-q^2)\frac{dP_l^m(q)}{dq}(-a\sin m\phi + b\cos m\phi)\ ,
\end{aligned}
\tag{38}
$$

where a, b are arbitrary constants.

For $m = 0$, with the use of the differential equation (36), (33) becomes

$$
\partial_q \ln \alpha(r,q) = -\frac{d}{dq}\ln\left((1-q^2)\frac{dP_l}{dq}\right)\ , \tag{39}
$$

which gives

$$
u = -\frac{(1-q^2)(dP_l/dq)}{lr^l}\ , \qquad v = \phi\ . \tag{40}
$$

Since the function v does not depend upon the parameter l, the sum of any number of u's with different l's also becomes a solution of (28) with the corresponding sum of $G_l^0 = P_l(q)/r^{l+1}$. Indeed this solution satisfies the ansatz equation (19), although it contains terms with different homogeneities. This class of solutions of the ansatz equation is also obtained by taking z-derivatives of the $l = 0$ solution (25),

$$
G = \sum_n a_n\,\partial_z^n\frac{1}{r}\ , \quad u = \sum_n a_n\,\partial_z^n\frac{z}{r}\ , \quad v = \phi = \arctan\frac{y}{x}\ , \tag{41}
$$

where a_n are constants. These solutions also belong to the space of $m = 0$ solutions.

To find an explicit form of $T(q)$ in (34) for $m \neq 0$ cases, let us introduce the following identity,

$$
(1-q^2)\frac{dP_l^m}{dq} = (l+m)P_{l-1}^m - lqP_l^m\ , \tag{42}
$$

and a series expansion formula for P_l^m,

$$
\begin{aligned}
P_l^m(q) &= (1-q^2)^{\frac{m}{2}}\frac{d^m}{dq^m}P_l(q) \\
&= (1-q^2)^{\frac{m}{2}}\sum_{r=0}^{[(l-m)/2]}\frac{(-1)^r}{2^r\,r!}\frac{(2l-2r-1)!!}{(l-m-2r)!}\,q^{l-m-2r}\ .
\end{aligned}
\tag{43}
$$

Using these formulae, we can rewrite the integrand of $T(q)$ as a ratio of polynomials with respect to q,

$$
t(q) = \frac{l(l+1)\,P_l^m}{(1-q^2)dP_l^m/dq} = \frac{l(l+1)\,P_l^m(q)}{(l+m)P_{l-1}^m - lqP_l^m}\ . \tag{44}
$$

Note that the factor $(1-q^2)^{\frac{m}{2}}$ in the RHS of (43) does not appear in the above ratio. The integral may be performed in principle by resolving the integrand into partial fractions.

In the following, we demonstrate explicit solutions of the ansatz equation (19) for $m = l$ to $m = l-4$.

[1] $m = l$

$$G = \frac{(\sin\theta)^l}{r^{l+1}}\, e^{il\phi} , \qquad u = \frac{(\cos\theta)^{l+1}}{l\, r^l} , \quad v = -i(\tan\theta)^l\, e^{il\phi} . \tag{45}$$

[2] $m = l-1$

$$\begin{aligned} G &= \frac{(\sin\theta)^{l-1}\cos\theta}{r^{l+1}}\, e^{i(l-1)\phi} , \\ u &= \frac{(l\cos^2\theta-1)^{\frac{l+1}{2}}}{l\, r^l} , \quad v = -\frac{i}{(l-1)}\frac{(\sin\theta)^{l-1}}{(l\cos^2\theta-1)^{\frac{l-1}{2}}}\, e^{i(l-1)\phi} . \end{aligned} \tag{46}$$

[3] $m = l-2$

$$\begin{aligned} G &= \frac{(\sin\theta)^{l-2}((2l-1)\cos^2\theta-1)}{r^{l+1}}\, e^{i(l-2)\phi} , \\ u &= \frac{1}{l\, r^l}(\cos\theta)^{\frac{l(l+1)}{5l-4}}((2l-1)l\cos^2\theta-(5l-4))^{2\frac{(l+1)(l-1)}{5l-4}} , \\ v &= -\frac{i}{(l-2)}\frac{(\sin\theta)^{l-2}(\cos\theta)^{-\frac{(l-2)^2}{5l-4}}}{((2l-1)l\cos^2\theta-(5l-4))^{\frac{(2l-1)(l-2)}{5l-4}}}\, e^{i(l-2)\phi} . \end{aligned} \tag{47}$$

[4] $m = l-3$

$$\begin{aligned} G &= \frac{(\sin\theta)^{l-3}\cos\theta((2l-1)\cos^2\theta-3)}{r^{l+1}}\, e^{i(l-3)\phi} , \\ u &= \frac{1}{l\, r^l}\frac{(\omega+9(l-1)-2l(2l-1)\cos^2\theta)^{(l+1)\frac{3(l-3)+\omega}{4\omega}}}{(\omega-9(l-1)+2l(2l-1)\cos^2\theta)^{(l+1)\frac{3(l-3)-\omega}{4\omega}}} , \\ v &= i\frac{e^{i(l-3)\phi}(\sin\theta)^{l-3}}{4l(l-3)(2l-1)}\frac{(\omega-9(l-1)+2l(2l-1)\cos^2\theta)^{(l-3)\frac{3(l+1)-\omega}{4\omega}}}{(\omega+9(l-1)-2l(2l-1)\cos^2\theta)^{(l-3)\frac{3(l+1)+\omega}{4\omega}}} , \end{aligned} \tag{48}$$

where $\omega = \sqrt{3(19l^2-50l+27)}$.

[5] $m = l - 4$

$$\begin{aligned}
G &= \frac{(\sin\theta)^{l-4}((2l-1)(2l-3)\cos^4\theta - 6(2l-3)\cos^2\theta + 3)}{r^{l+1}}\, e^{i(l-4)\phi}\ , \\
u &= \frac{1}{l\, r^l}(\cos\theta)^{\frac{l(l+1)}{9l-16}} \frac{(l(2l-1)\cos^2\theta - (7l-8) - 2\xi)^{\frac{2(l+1)(l-2)(l-4+\xi)}{(9l-16)\xi}}}{(l(2l-1)\cos^2\theta - (7l-8) + 2\xi)^{\frac{2(l+1)(l-2)(l-4-\xi)}{(9l-16)\xi}}}\ , \qquad (49) \\
v &= -i\frac{(2l-3)}{l(l-4)(2l-1)}\,(\sin\theta)^{l-4}(\cos\theta)^{-\frac{(l-4)^2}{9l-16}} \times \\
&\quad \times \frac{(l(2l-1)\cos^2\theta - (7l-8) + 2\xi)^{\frac{2(l+1)(l-2)(l-4-\xi)}{(9l-16)\xi}+1}}{(l(2l-1)\cos^2\theta - (7l-8) - 2\xi)^{\frac{2(l+1)(l-2)(l-4+\xi)}{(9l-16)\xi}-1}}\, e^{i(l-4)\phi}\ ,
\end{aligned}$$

where $\xi = \sqrt{(l-2)(11l^2 - 40l + 24)/(2l-3)}$.

5 Fourier mode solutions

The equations of motion (5) in the case of two fields can be recast as, in general dimension,

$$\left(\frac{\partial v}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - \frac{\partial v}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\right)^2 u \;=\; 0\ , \qquad (50)$$

$$\left(\frac{\partial u}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - \frac{\partial u}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\right)^2 v \;=\; 0\ . \qquad (51)$$

Suppose now we look for a solution of the form

$$u \;=\; A\exp(k_\nu x^\nu)\ , \quad v \;=\; B\exp(p_\nu x^\nu) \;+\; C\exp(q_\nu x^\nu)\ , \qquad (52)$$

where $A,\ B,\ C$ are constants, and $\vec{k},\ \vec{p},\ \vec{q}$ are constant d-dimensional vectors. (If C=0, then this is simply a special case of the form discussed earlier.) Then the coefficient of AB^2 in (50) is

$$\begin{aligned}
&\exp(p_s x^s)\left(p_\mu\frac{\partial}{\partial x^\nu} - p_\nu\frac{\partial}{\partial x^\mu}\right)(p_\mu k_\nu - p_\nu k_\mu)\exp(p_r x^r + k_r x^r) \\
&= \exp(2p_r x^r + k_r x^r)\left(p^2k^2 - (p\cdot k)^2\right) \;=\; 0\ . \qquad (53)
\end{aligned}$$

Similarly, the coefficient of AC^2 in (50) vanishes, provided

$$q^2k^2 - (q\cdot k)^2 \;=\; 0\ . \qquad (54)$$

These conditions ensure that (51) is also satisfied without any further condition. They have only nontrivial complex solutions in Euclidean space, but have real (or pure imaginary, as befits plane wave solutions) in Minkowski space. What is remarkable is that the cross term proportional to ABC in (50) also vanishes if those two conditions (53),(54) are imposed. The vanishing of this term is tantamount to the following condition;

$$(p_\mu(q_\nu+k_\nu)-p_\nu(q_\mu+k_\mu))(q_\mu k_\nu-q_\nu k_\mu)+(q_\mu(p_\nu+k_\nu)-q_\nu(p_\mu+k_\mu))(p_\mu k_\nu-p_\nu k_\mu)=0\ . \tag{55}$$

Simplifying this equation by using the previous conditions, we have

$$(p\cdot q)[(p+q+2k)\cdot k]-(p\cdot k)q^2-(q\cdot k)p^2-2(p\cdot k)(q\cdot k)\ =\ 0\ . \tag{56}$$

This is satisfied if the pair $(p,\ q)$ obeys a similar restriction to $(p,\ k)$ and $(q,\ k)$. These restrictions enable (56) to be expressed in the form

$$(p\cdot q-|p||q|)(2|k|+|p|+|q|)\ =\ 0\ . \tag{57}$$

which is clearly satisfied if $p\cdot q$ is replaced by $|p||q|$. By a linear transformation, $k,\ p,\ q$ may be represented by three-dimensional vectors. $\vec{k}$ is then expressed as a linear combination,

$$\vec{k}\ =\ \alpha\vec{p}+\beta\vec{q}+\gamma\vec{p}\times\vec{q}\ . \tag{58}$$

Taking scalar products of both sides with respect to $\vec{p}$ and $\vec{q}$ yields the required condition;

$$\vec{p}\cdot\vec{q}=|p||q|\ . \tag{59}$$

With the metric $(+,\ -,\ -)$, the vectors may be parametrised by

$$\begin{aligned}
\vec{k}\ &=\ (\sinh\theta,\ \cosh\theta,\ 0)\ ,\\
\vec{p}\ &=\ (a_1,\ a_2,\ a_1\cosh\theta-a_2\sinh\theta)\ ,\\
\vec{q}\ &=\ (b_1,\ b_2,\ b_1\cosh\theta-b_2\sinh\theta)\ .
\end{aligned} \tag{60}$$

These vectors all satisfy the condition that the cross product of any pair are lightlike automatically. Further terms may also be added to both u and v, provided they fall within this parametrisation; all pairs have lightlike cross products, so the modes which fall within this class satisfy a superposition principle. It is expected that the analysis can be extended at the cost of more technical complexity, to the case of three or more fields. The surprising result is that although the equations are non-linear, nevertheless a large class of plane wave solutions obey a superposition principle. Note further that the analysis extends to the situation where both fields are linear combinations of plane wave solutions; Suppose

$$u=\sum_j A_j\exp(p_j\cdot x)\ ,\qquad v=\sum_j B_j\exp(q_j\cdot x)\ . \tag{61}$$

Then apart from the trivial case where u and v are functionally related, the conditions for a solution of this type are simply

$$(p_i \cdot p_j)^2 - p_i^2 p_j^2 \;=\; (k_i \cdot k_j)^2 - k_i^2 k_j^2 \;=\; (p_i \cdot k_j)^2 - p_i^2 k_j^2 \;=\; 0 \;, \quad \text{for } \forall \; (i,j) \;. \tag{62}$$

An important feature of a solution of this type is that the coefficients $A_j,\; B_j$ are all independent, and therefore may be treated as creation or annihilation operators in quantising the theory.

6 Incorporation of mass

A natural way to introduce a mass will be through the introduction of a term $M^{2n} \prod_1^n (\phi^j)^2$ into the Lagrangian. Then in the case of two fields in general dimension discussed in section 3, the conditions for a solution of the form of (10) require that $f,\; g$ are pure exponentials and (11) is modified to

$$(a_\rho a_\rho)(b_\sigma b_\sigma) \;-\; (a_\rho b_\rho)^2 \;=\; M^4 \;. \tag{63}$$

It seems that the incorporation of mass drastically reduces the variety of tractable solutions.

7 Discussion

In the case of two fields, in section 3, we have exhibited an arbitrary family of solutions to the second order equations of motion. These solutions are complex in Euclidean space, but real in Minkowski space. On the other hand, in section 4, we have demonstrated real solutions of the first order ansatz equation in Euclidean space. This situation is reminiscent of the instanton construction, as the self-dual ansatz for Yang-Mills also yields solutions which are real in Euclidean space. The solutions we have found are associated with spherical harmonics, labelled by two integers, which also brings to mind the fact that instantons also have a label, the instanton number.

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